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The Idea of Infinite-Volume Limit for Some Notions of Mixing in Infinite Ergodic Theory

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One of the most important signatures of chaos in a dynamical system is the decay of the correlations between its observables. In the prototypical case of a map $T : \mathcal{M} \longrightarrow \mathcal{M}$, preserving a probability measure μ, this reads:

$$\lim_{n\to\infty} \int_{\mathcal{M}} (f \circ T^n) g \, d\mu = \int_{\mathcal{M}} f \, d\mu \cdot \int_{\mathcal{M}} g \, d\mu, \tag{1}$$

for every pair of *observables* (i.e., square-integrable functions) f, g in phase space. It is no loss of generality to require the above only for indicator functions, which is tantamount to asking that

$$\lim_{n\to\infty} \mu(T^{-n} A \cap B) = \mu(A)\mu(B), \tag{2}$$

for all measurable sets $A, B \subset \mathcal{M}$. It is because of (2), and the pictures that illustrated this definition in the classical textbooks of ergodic theory, that the above property came to be called *mixing*.

Infinite ergodic theory is the discipline that studies the stochastic properties of dynamical systems in infinite-measure spaces. Often, as is the case here, it is assumed that the dynamical system in question preserves an infinite (σ-finite) measure.

Transporting the fundamental—but intrinsically probabilistic—notion of mixing to the realm of infinite ergodic theory has long been an open problem. References to this issue can be found in the literature at least as far back as 1937, when Hopf devoted a section of his famous *Ergodentheorie* [1] to an example of a dynamical system that he calls 'mixing'. It consists of a set $\mathcal{M} \subset \mathbb{R}^2$, of infinite Lebesgue measure, and a map $T : \mathcal{M} \longrightarrow \mathcal{M}$ preserving μ, the Lebesgue measure on $\mathcal{M}$. He proved a property that is equivalent to this one: there exists a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of positive numbers such that

$$\lim_{n\to\infty} \rho_n \, \mu(T^{-n} A \cap B) = \mu(A)\mu(B) \tag{3}$$

for all squarable sets $A, B \subset \mathcal{M}$ (a squarable set is a bounded set whose boundary has measure zero).

The quest for a universally agreed-upon definition of mixing in infinite ergodic theory (*infinite mixing* for short) became pressing by the late 1960's. In 1967, Krickeberg [2] proposed precisely (3) as the definition of mixing for a general class of dynamical systems (namely, almost-everywhere continuous endomorphisms of a σ-finite Borel space $(\mathcal{M}, \mu)$, with some additional mild technical conditions). Krickeberg's definition has been studied and applied several times since then but has failed to establish itself as the ultimate definition of infinite mixing. In this author's opinion, this is not so much because of the less-than-perfect requirement of a topological structure in a measure-theoretic problem, but rather for its inherent inability to describe the global, infinite-measure, aspects of a dynamics: after all, (3) only involves finite-measure sets.

In 1969, Krengel and Sucheston [3] approached the problem from a decisively more abstract point of view. They considered a measure-theoretic condition that is equivalent to finite mixing, and extended it, in a couple of ways, to systems endowed with an infinite (not necessarily invariant) measure. They named these two definitions, respectively, 'mixing' and 'complete mixing'. The authors themselves, however, proved results that imply that many reasonable (including all invertible) measure-preserving maps cannot be completely mixing. As for mixing, again specializing to measure-preserving maps, their definition is equivalent to

$$\lim_{n\to\infty} \mu(T^{-n} A \cap B) = 0, \tag{4}$$

for all finite-measure sets A, B. This is a rather brutal weakening of (3): for instance, it would classify a translation in $\mathbb{R}^d$ as mixing!

This history led Aaronson to write, in 1997 [4], that *"there is no reasonable generalisation of mixing."* Be that as it may, the problem remains of devising at least a concept—if a general and rigorous definition seems unattainable—that somehow expresses the tendency of many infinite-measure systems to "mix up" trajectories and lose memory of the initial state.

The shortcoming of the definitions recalled earlier was that none did quite capture the global aspects of the dynamics. To fix this, we introduce a new class of observables, called *global observables.* This will be a family, defined on a case-by-case basis, of bounded functions F for which the following limit exists:

$$\overline{\mu}(F) := \lim_{V \nearrow \mathcal{M}} \frac{1}{\mu(V)} \int_V F \, d\mu. \tag{5}$$

Here V is taken from a specified class of ever-larger finite-measure sets that cover the whole of $\mathcal{M}$, and the limit can be rigorously defined; $\overline{\mu}(F)$ is called the *infinite-voume average* of F. The idea is that a global observable should be supported throughout the phase space and look more or less the same in all large regions of it. On the opposite end, we call *local observables* a certain subspace, again specified in each case, of the integrable functions.

At this point, two notions of mixing emerge naturally. Skipping many necessary details, we say that we have *global-global* mixing if

$$\lim_{n\to\infty} \overline{\mu}((F \circ T^n)G) = \overline{\mu}(F)\overline{\mu}(G), \tag{6}$$

for any two global observables F, G; and we have *global-local* mixing if

$$\lim_{n\to\infty} \mu((F \circ T^n)g) = \overline{\mu}(F)\mu(g), \tag{7}$$

for any global F and local g (with the notation $\mu(g) := \int g \, d\mu$).

Once all the objects described above have been exactly specified for a system of interest, (6)-(7) are turned into a number of rigorous definitions (we write two versions of global-global mixing and three versions of global-local mixing). We verify all or some of these definitions for a few examples of dynamical systems preserving an infinite-measure, such as: random walks in $\mathbb{Z}^d$, uniformly expanding discontinuous maps of $\mathbb{R}$, non-uniformly expanding intermittent maps with an indifferent fixed point, aperiodic Lorentz gases.

References

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[4] J. Aaronson, *An introduction to infinite ergodic theory* (MSM 50, American Mathematical Society, Providence, RI, 1997).